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An Initial Value Method for the Computation of the Characteristic Values and Functions of an Integral Operator-II

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A convergence proof is given for an initial value method for finding the spectral properties of an integral operator. The algorithm, which is based on a Cauchy system for the Fredholm determinants, is related to the Nyström method and results of Anselone and Atkinson become applicable. The proof is also shown to work for a modification of the procedure due to Kalaba and Scott.

1. INTRODUCTION

Recently a new initial value method for computing the characteristic values and functions of an integral operator has been developed [1]. The technique is based on solving a Cauchy problem for a set of integrodifferential equations for the Fredholm determinants of the given kernel. The work done up till now has consisted of the development of numerically feasible algorithms, ignoring the question of convergence [1, 2, 3]. In this note we will fill this gap by relating the method to the classical procedure based on linear algebraic equations and then applying recent work by Atkinson, Anselone and Moore on collectively compact operators.

The development relies heavily on the author's previous papers [1, 4, 5] and we will refer the reader there for many of the details of the proofs.

2. NOTATION

For a given matrix A det A will denote its determinant and adj A its classical adjoint. I will be used for either the $N \times N$ identity matrix or the identity operator on a given vector space. Matrices will be indicated by capitols and the notation $[A]_{ij}$ will be used to denote the ijth element of A.

Subscript notation will be used for derivatives.

3. The Numerical Algorithm

Consider the problem of finding the characteristic values and functions of the continuous kernel $k(t, s)$, $a \leqslant (t, s) \leqslant b$; that is, we want to compute λ_c and $u_c(t) \not\equiv 0$ satisfying

$$u_c(t) = \lambda_c \int_a^b k(t, s)\, u_c(s)\, ds. \tag{1}$$

It is well known that the numbers λ_c are the zeros of the Fredholm determinant $d(\lambda)$ and that under appropriate conditions the characteristic functions $u_c(t)$ can be obtained from the Fredholm minor $D(t, s, \lambda)$ [6]. In [1] it was shown that the pair $(d(\lambda), D(t, s, \lambda))$ satisfied the initial value problem

$$\begin{aligned} d(\lambda)\, D_\lambda(t, s, \lambda) = {} & -\left(\int_a^b D(s, s, \lambda)\, ds\right) D(t, s, \lambda) \\ & + \int_a^b D(t, \tau, \lambda)\, D(\tau, s, \lambda)\, d\tau, \end{aligned} \tag{2}$$

$$d_\lambda(\lambda) = -\int_a^b D(s, s, \lambda)\, ds, \tag{3}$$

$$D(t, s, 0) = k(t, s), \qquad d(0) = 1. \tag{4}$$

Integration of (2)–(4) enables one to compute $d(\lambda)$ and $D(t, s, \lambda)$, obtaining λ_c when $d(\lambda_c) = 0$ and $u_c(t)$ from $D(t, s, \lambda_c)$ for fixed s. Numerically this is accomplished by discretizing the right-hand side using a quadrature rule of the form

$$\int_a^b f(s)\, ds \simeq \sum_{i=1}^N w_i f(s_i), \tag{5}$$

giving

$$\begin{aligned} \hat{d}(\lambda)\, \hat{D}_\lambda(t, s, \lambda) = {} & -\left(\sum_{k=1}^N w_k \hat{D}(s_k, s_k, \lambda)\right) \hat{D}(t, s, \lambda) \\ & + \sum_{k=1}^N w_k \hat{D}(t, s_k, \lambda)\, \hat{D}(s_k, s, \lambda), \end{aligned} \tag{6}$$

$$\hat{d}_\lambda(\lambda) = -\sum_{k=1}^N w_k \hat{D}(s_k, s_k, \lambda), \tag{7}$$

$$\hat{D}(t, s, 0) = k(t, s), \qquad \hat{d}(0) = 1. \tag{8}$$

Letting $D_{ij}(\lambda) = \hat{D}(t_i, s_j, \lambda)$, $(i, j = 1, 2,..., N)$ we see that $D_{ij}(\lambda)$ and $\hat{d}(\lambda)$ satisfy the set of ordinary differential equations

$$\hat{d}(\lambda)\, D_{ij,\lambda}(\lambda) = -\left(\sum_{k=1}^{N} w_k D_{kk}(\lambda)\right) D_{ij}(\lambda) + \sum_{k=1}^{N} w_k D_{ik}(\lambda)\, D_{kj}(\lambda), \tag{9}$$

$$\hat{d}_{\lambda}(\lambda) = -\sum_{k=1}^{N} w_k D_{kk}(\lambda), \tag{10}$$

$$D_{ij}(0) = k(t_i, s_j), \qquad \hat{d}(0) = 1, \qquad (i, j = 1, 2,..., N). \tag{11}$$

Integrating the system (9)–(11) constitutes the numerical method for obtaining λ_c and $u_c(t)$.

Numerical results for this and several variants of the above algorithm have been presented in the papers [1, 2, 4].

In order to establish convergence of the method based on the equations (9)–(11) we will relate them to the system (6)–(8). In fact, it will be shown that knowledge of $D_{ij}(\lambda)$, $(i, j = 1, 2,..., N)$ and $\hat{d}(\lambda)$ is sufficient to determine $\hat{D}(t, s, \lambda)$.

Define matrices K and W by $K = \{k(t_i, s_j)\}$, $W = \{w_i \delta_{ij}\}$ where δ_{ij} is the Kronecker delta.

THEOREM 2.1. *There is a unique solution to* (9)–(11) *given by*

$$D_{ij}(\lambda) = [\operatorname{adj}(I - \lambda KW)\, K]_{ij} \qquad \textit{and} \qquad \hat{d}(\lambda) = \det(I - \lambda KW).$$

Proof. This was established in [4]. See [4] for details.

THEOREM 2.2. *The initial value problem* (6)–(8) *has a unique solution for all complex* λ *and* $(t, s) \in [a, b] \times [a, b]$. *Furthermore,*

$$\hat{D}(t_i, s_j, \lambda) = D_{ij}(\lambda), \qquad (i, j = 1, 2,..., N).$$

Proof. Uniqueness is established first. Observe that if (6)–(8) has a solution $\hat{d}(\lambda)$ is differentiable and thus continuous at $\lambda = 0$. Since $d(0) = 1$ there is a circle $|\lambda| < \epsilon$ such that for $|\lambda| < \epsilon$, $\hat{d}(\lambda) \neq 0$. Thus for $|\lambda| < \epsilon$ we may divide (6) and (7) by $\hat{d}(\lambda)$ and consider the function

$$\hat{R}(t, s, \lambda) = \hat{D}(t, s, \lambda)/\hat{d}(\lambda).$$

It is easily shown using (5), (6) and (7) that $\hat{R}(t, s, \lambda)$ satisfies the initial value problem [5]

$$\hat{R}_\lambda(t, s, \lambda) = \sum_{k=1}^{N} w_k \hat{R}(t, s_k, \lambda)\, \hat{R}(s_k, s, \lambda), \tag{12}$$

$$\hat{R}(t, s, 0) = k(t, s). \tag{13}$$

In [5] it was shown that (12–13) has a unique solution given by

$$\hat{R}(t, s, \lambda) = k(t, s) + \lambda \sum_{k=1}^{N} w_k k(t, s_k)\, \hat{r}(s_k, s, \lambda), \tag{14}$$

where

$$\hat{r}(s_k, s, \lambda) = \sum_{k=1}^{N} w_k [(I - \lambda KW)^{-1}]_{ik}\, k(s_k, s), \tag{15}$$

for $|\lambda| < |\lambda_1|$ where λ_1 is the characteristic value of minimum modulus of the matrix KW. Therefore for $|\lambda| < \min(\epsilon, |\lambda_1|)$

$$\hat{D}(t, s, \lambda) = \hat{d}(\lambda)\, \hat{R}(t, s, \lambda).$$

Now it is easily seen from (9)–(11) that $\hat{D}(t_i, s_j, \lambda)$ and $\hat{d}(\lambda)$ satisfy (6)–(8). By Theorem 2.1 $\hat{D}(t_i, s_j, \lambda) = D_{ij}(\lambda)$ and so $\hat{d}(\lambda) = \det(I - \lambda KW)$. Thus for $|\lambda| < \min(\epsilon, |\lambda_1|)$ $\hat{d}(\lambda)$ and $\hat{D}(t, s, \lambda)$ are uniquely determined. Since they are obviously entire functions of λ, standard continuation arguments establish uniqueness for all λ.

To prove existence we simply use the formulas $\hat{d}(\lambda)$ and $\hat{D}(t, s, \lambda)$ given above. Q.E.D.

4. A Discrete Fredholm Theory

To proceed we must establish some further properties of $\hat{d}(\lambda)$ and $\hat{D}(t, s, \lambda)$. In fact, we will show that these quantities give a Fredholm theory for a discrete operator associated with the kernel $k(t, s)$.

Let C denote the Banach space of continuous functions on $[a, b]$ equipped with the norm

$$\| x(t) \| = \sup_{t \in [a,b]} | x(t) |, \qquad x(t) \in C.$$

Let $T: C \to C$ be defined by

$$(Tx)(t) = \int_a^b k(t, s)\, x(s)\, ds. \tag{16}$$

Denote by φ_N the quadrature rule given by (5). Corresponding to φ_N define the operator $T_N: C \to C$ by

$$(T_N x)(t) = \sum_{k=1}^{N} w_k k(t, s_k)\, x(s_k). \tag{17}$$

Both T and T_N are compact operators on C and T_N is of finite rank [7].

THEOREM 3.1. *The characteristic values $\hat{\lambda}$ of T_N are given as the solutions of $\hat{d}(\lambda) = 0$. If $\hat{D}(t, s, \hat{\lambda}) \not\equiv 0$ then for fixed s it is a characteristic function of T_N for $\hat{\lambda}$.*

Proof. Let $\hat{\lambda}$ be a characteristic value for T_N. Then

$$u(t) = \hat{\lambda} \sum_{k=1}^{N} w_k k(t, s_k)\, u(s_k). \tag{18}$$

Note that at least one $u(s_k)$ is not zero, since if not then by (18) $u(t)$ would be zero for all $t \in [a, b]$ and thus not a characteristic function. Let $t = t_j$, $j = 1, 2,..., N$ we see that $u_j \equiv u(t_j)$, $j = 1, 2,..., N$ satisfies

$$u_j = \hat{\lambda} \sum_{k=1}^{N} k(t_j, s_k)\, u(s_k)\, w_k . \tag{19}$$

Thus $\hat{\lambda}$ is a characteristic value of KW and so $\det(I - \hat{\lambda} KW) = 0 = d(\hat{\lambda})$.

Now assume that $d(\hat{\lambda}) = 0$. Then $\hat{\lambda}$ is a characteristic value of KW. Let u_j, $j = 1, 2,..., N$ be a characteristic vector for $\hat{\lambda}$. Define $u(t)$ by

$$u(t) = \hat{\lambda} \sum_{k=1}^{N} w_k(t, s_k)\, u_k . \tag{20}$$

Then

$$\begin{aligned}
[T_N u](t) &= \sum_{k=1}^{N} w_k k(t, s_k)\, u(s_k) \\
&= \sum_{k=1}^{N} w_k k(t, s_k) \left[\hat{\lambda} \sum_{j=1}^{N} k(s_k, \tau_j)\, u_j\right] \\
&= \sum_{k=1}^{N} w_k k(t, s_k)\, u_k
\end{aligned}$$

and so $u(t)$ is a characteristic function for T_N corresponding to the characteristic value $\hat{\lambda}$.

To obtain the result on characteristic functions use the fact, proved in [5], that $\hat{R}(t, s, \lambda)$ satisfies

$$\hat{R}(t, s, \lambda) = k(t, s) + \lambda \sum_{k=1}^{N} w_k k(t, s_k)\, \hat{R}(s_k, s, \lambda). \tag{21}$$

Multiply both sides of (21) by $\hat{d}(\lambda)$ giving

$$\hat{D}(t, s, \lambda) = k(t, s)\, \hat{d}(\lambda) + \lambda \sum_{k=1}^{N} w_k k(t, s_k)\, \hat{D}(s_k, s, \lambda). \tag{22}$$

Let $\lambda \to \hat{\lambda}$ in (22) and the fact that $\hat{d}(\hat{\lambda}) = 0$ to get

$$\hat{D}(t, s, \hat{\lambda}) = \hat{\lambda} \sum_{k=1}^{N} w_k k(t, s_k)\, \hat{D}(s_k, s, \hat{\lambda}). \tag{23}$$

This proves the assertion of the theorem. Q.E.D.

Note that if a characteristic value $\hat{\lambda}$ is simple then $\hat{D}(t, s, \hat{\lambda}) \not\equiv 0$; in this case we always obtain nontrivial characteristic functions by the above algorithm. In fact, since $\hat{d}_\lambda(\hat{\lambda}) \neq 0$ and $d_\lambda(\hat{\lambda}) = -tr(D(\hat{\lambda})\, W)$ [4], where $D(\lambda) = \{D_{ij}(\lambda)\}$, then $D(\hat{\lambda})\, W \not\equiv 0$. Thus $D(\hat{\lambda}) \not\equiv 0$ and so $\hat{D}(t, s, \hat{\lambda}) \not\equiv 0$.

5. Convergence

Our convergence proof will be based on the results obtained above and those of Atkinson and Anselone on collectively compact operators [7]. To avoid undue complications we will make some additional assumptions concerning the spectrum of T.

Since T is compact there are at most a countable number of characteristic values having accumulation points at most at ∞. In addition the multiplicity of each characteristic value is finite. We will assume from now on that the dimension of each characteristic subspace is one.

Now consider a sequence of quadrature rules $\{\varphi_N\}$ of the form (5). Assume that

$$\lim_{N\to\infty} \varphi_N(f) = \lim_{N\to\infty} \sum_{i=1}^{N} w_{iN} f(s_{iN}) = \int_a^b f(t)\, dt.$$

Let $\{T_N\}$ be the sequence of operators corresponding to $\{\varphi_N\}$. It then follows that $\{T_N\}$ is collectively compact [7] and that $T_N \to T$ pointwise.

Let $\sigma(T) = \{\lambda \in \not{C} \mid \lambda - T \text{ is one-to-one}\}$ be the spectrum of T and let $\sigma(T_N)$ be defined similarly for T_N. Then it is well known that the characteristic values of T and T_N are the reciprocals of the nonzero elements in $\sigma(T)$ and $\sigma(T_N)$, respectively. From Theorem 4.16 of [7] it follows that for N sufficiently large that the dimension of the characteristic subspaces of T_N is one and therefore that the zeros of $\hat{d}(\lambda)$ are simple. As a consequence, the theory of Section 4 applies, and for all N large enough approximate characteristic vectors are obtained from $\hat{D}(t, \tau, \hat{\lambda})$. Now let N be large enough to satisfy the above conditions and let $C(T_N) = \{\lambda_{N1}, \lambda_{N2}, ..., \lambda_{NN}\}$ be the characteristic values of T_N. If λ_c is a characteristic value of T then Theorem 4.17 of [7] says that there exists a sequence $\{\lambda_{Nk_N}\}$ $(\lambda_{Nk_N} \subset C(T_N))$ such that

$$\lim_{N\to\infty} \lambda_{Nk_N} = \lambda_c .$$

Now relabel $\{\lambda_{Nk_N}\}$ as $\{\lambda_N\}$ so that $\lambda_N \to \lambda_c$. Let $x_N(t)$ be a characteristic function for λ_N obtained from $\hat{D}(t, s, \lambda_N)$ for fixed s. Define $u_N(t)$ by $u_N(t) = x_N(t)/\| x_N(t)\|$ so that $\| u_N(t)\| = 1$. By Theorem 4.11 of [7] there exists a subsequence $\{u_{Ni}(t)\}$ such that $u_{Ni}(t) \to u(t)$ where $u(t) = \hat{\lambda}_c T u(t)$, $u(t) \neq 0$, is a characteristic function for λ_c. To summarize then, we have proved the following theorems.

THEOREM 5.1. *Let T have simple characteristic values. If λ_c is a characteristic value of T then for N sufficiently large the roots of $\hat{d}(\lambda)$ are simple and there exists a sequence $\{\lambda_{Nk_N}\}$ $(\lambda_{Nk_N} \subset C(T_N))$ such that*

$$\lim_{N\to\infty} \lambda_{Nk_N} = \lambda_c .$$

THEOREM 5.2. *Let*

$$x_N(t) = \hat{D}(t, s, \lambda_{Nk_N})/\| \hat{D}(t, s, \lambda_{Nk_N})\|$$

where t is fixed. Then there exists a subsequence $\{x_{Ni}(t)\}$ such that

$$\lim_{N_i\to\infty} \| x_{Ni}(t) - x(t)\| = 0$$

where $\lambda_c T x(t) = x(t)$, $x(t) \neq 0$.

Theorems 4.1 and 4.2 establish the convergence of the algorithm presented in [1, 4].

6. DISCUSSION

The convergence proof presented above establishes the convergence of the algorithm first presented in [1]. In a recent series of papers [2, 3, 8] Kalaba and Scott have developed a modification of the method based on Eqs. (2)–(4) which avoids the necessity of integrating over the zeros of $d(\lambda)$ and integrates *around* them in the complex plane. We wish to show that the approximate characteristic functions and values obtained by their procedure equal the ones obtained from (5)–(8). As a consequence the convergence proofs given in Sections 4 and 5 apply.

Let λ_c be a simple zero of $d(\lambda)$. Since $d(\lambda)$ is an entire function of λ [6], Cauchy's theorem gives

$$\lambda_c = \frac{1}{2\pi i} \int_C \lambda \frac{d_\lambda(\lambda)}{d(\lambda)}\, d\lambda, \tag{24}$$

where C is a contour in the complex plane having λ_c in its interior and such that no other zeros of $d(\lambda)$ lie on or inside C. Using (10) in (24)

$$\lambda_c = \frac{1}{2\pi i} \int_C \lambda \left[- \int_a^b \frac{D(s, s, \lambda)\, ds}{d(\lambda)} \right] d\lambda. \tag{25}$$

Discretizing the integral in (25) gives the value

$$\bar{\lambda} = \frac{-1}{2\pi i} \int_C \lambda \, \frac{\sum_{k=1}^{N} w_k \hat{D}(s_k\,, s_k\,, \lambda)}{\hat{d}(\lambda)}\, d\lambda. \tag{26}$$

Kalaba and Scott compute $\bar{\lambda}$ by integrating (9)–(11) from 0 to a point P on C. At P the differential equation

$$\frac{d\psi}{d\lambda}(\lambda) = \frac{-1}{2\pi i} \sum_{k=1}^{N} \lambda w_k \frac{\hat{D}(s_k\,, s_k\,, \lambda)}{\hat{d}(\lambda)}, \tag{27}$$

is adjoined with initial condition $\psi(P) = (1/2\pi i)\, P(\hat{d}_\lambda(P)/\hat{d}(P))$. (9)–(11) and (27) are now integrated around C and the increment $\Delta\psi$ in $\psi(\lambda)$ is $\bar{\lambda}$.

Theorem 2.1 shows that $\hat{d}(\lambda)$ and $D(\lambda)$ are entire functions (in fact, polynomials in λ). Application of Cauchy's theorem to $\hat{d}(\lambda)$ gives

$$\begin{aligned} \hat{\lambda} &= \frac{1}{2\pi i} \int_C \lambda \, \frac{\hat{d}_\lambda(\lambda)}{\hat{d}(\lambda)}\, d\lambda \\ &= -\frac{1}{2\pi i} \int_C \lambda \, \frac{[\Sigma w_k \hat{D}(s_k\,, s_k\,, \lambda)]}{\hat{d}(\lambda)}\, d\lambda. \end{aligned} \tag{28}$$

Comparing (26) and (28) shows that $\bar{\lambda} = \hat{\lambda}$. The application of Cauchy's formula to $D_{ij}(\lambda)$ shows that

$$D_{ij}(\hat{\lambda}) = \frac{1}{2\pi i} \;{}_{C}\, \frac{D_{ij}(\lambda)}{\lambda - \hat{\lambda}}\, d\lambda, \tag{29}$$

so that $D_{ij}(\hat{\lambda})$ can be obtained by integrating the differential equation

$$\frac{d\Phi_{ij}(\lambda)}{d\lambda} = \frac{1}{2\pi i} \frac{D_{ij}(\lambda)}{\lambda - \hat{\lambda}}, \tag{30}$$

$$\Phi_{ij}(P) = \frac{1}{2\pi i} \frac{D_{ij}(P)}{P - \hat{\lambda}}, \tag{31}$$

around C. As above, this is easily seen to agree with the procedure given in [3].

The above discussion shows that the quantities computed by Kalaba and Scott agree with those computed by (9)–(11). Thus Theorems 5.1 and 5.2 apply and we obtain a convergence proof for their method also.

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